Basic Introduction to Quotient Geometry

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Outline

- Quotient space, Horizontal space, Vertical space.
- Horizontal lift and Horizontal curves.
- Quotient space Riemannian metrics and Geodesics.

Quotient space

$$\mathcal{M}_1/G = \mathcal{M}_2$$

$\mathcal{M}_2$ is the quotient space of $\mathcal{M}_1$ under a specified action by the group G.

What does this mean?

Quotient space (Ex 1: Space of concentric circles)

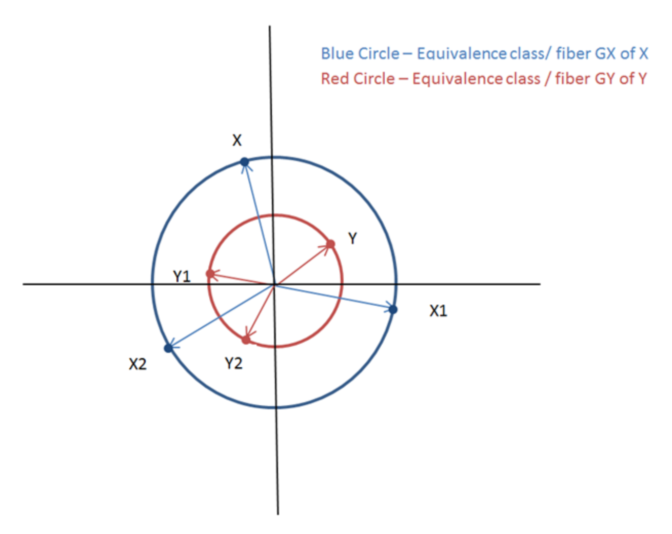
$G = \mathcal{O}_2 = \left\{ R \in \mathcal{R}^{2\times 2} \mid R^\top R = I_2 \right\}$
$\mathcal{M}_1 = \mathcal{R}^2$
Action: Left multiplication

Let $x \in \mathcal{M}_1$ and $g \in G$. The action of g on x produces $gx \in \mathcal{M}_1$. (because the action here is left multiplication)

The action of G on $x \in \mathcal{M}_1$ produces the set $[x] = Gx = \{gx \mid g \in G\}$. Note that gx is in $\mathcal{M}_1$ for all $g \in G$.

Every element in Gx is said to be equivalent to x under this group action. Gx is referred to as equivalence class of x (also referred to as fiber).

Quotient space (Ex 1: Space of concentric circles)



Quotient space (Ex 1: Space of concentric circles)

The action of G divides $\mathcal{M}_1$ into multiple disjoint equivalence classes (which happen to be circles centered at the origin in this example).

Quotient space $\mathcal{M}_2$ is nothing but the set of all equivalence classes. In this example, quotient space is the space of circles centered at the origin in $\mathcal{R}^2$.

Quotient space (Ex 2: Real Projective Space)

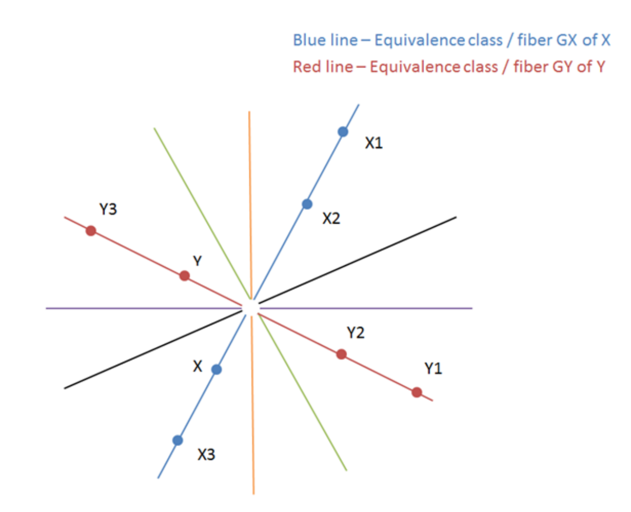
$G = \mathcal{R} - \{0\}$
$\mathcal{M}_1 = \mathcal{R}^2 - \{0\}$
Action: Scalar multiplication

$$\mathcal{R}^2 - \{0\}/\mathcal{R} - \{0\} = \mathcal{RP}^1$$

For a given $x \in \mathcal{M}_1$, its equivalence class is given by $Gx = \{gx \mid g \in \mathcal{R} - \{0\}\}$, which can be associated with a unique line(except the origin) in $\mathcal{R}^2$.

Real projective space $\mathcal{RP}^1$ is nothing but the space of lines passing through the origin in $\mathcal{R}^2$.

Quotient space (Ex 2: Real Projective Space)



Quotient space (Exercise 1)

$G = \mathcal{O}_3 = \left\{ R \in \mathcal{R}^{3\times 3} \mid R^\top R = I_3 \right\}$
$\mathcal{M}_1 = \mathcal{R}^3$
Action: Left multiplication

What is $\mathcal{M}_1 / G$?

Quotient space (Exercise 1)

$G = \mathcal{O}_3 = \left\{ R \in \mathcal{R}^{3\times 3} \mid R^\top R = I_3 \right\}$
$\mathcal{M}_1 = \mathcal{R}^3$
Action: Left multiplication

What is $\mathcal{M}_1/G$?

$\mathcal{M}_1/G$ is the space of spheres centered at the origin in $\mathcal{R}^3$.

Quotient space (Exercise 2)

$G = \mathcal{R} - \{0\}$
$\mathcal{M}_1 = \mathcal{R}^3 - \{0\}$
Action: Scalar multiplication

What is $\mathcal{M}_1/G$?

Quotient space (Exercise 2)

$G = \mathcal{R} - \{0\}$
$\mathcal{M}_1 = \mathcal{R}^3 - \{0\}$
Action: Scalar multiplication

What is $\mathcal{M}_1/G$?

$\mathcal{M}_1/G$ is the real projective space $\mathcal{RP}^2$, i.e., the space of lines passing through the origin in $\mathcal{R}^3$.

Quotient geometry

We have seen how some manifolds can be interpreted as quotient spaces of others manifolds.

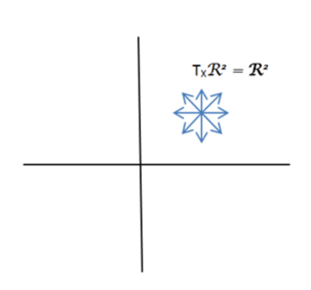
What is the use of this?

Given a manifold with closed form expressions for geodesics, parallel transport, distances, etc., we can easily (relatively) derive closed form expressions for these quantities on quotient spaces of that manifold.

Quotient geometry: Tangent space

Given a Riemannian manifold $\mathcal{M}$, at every point $X \in \mathcal{M}$, we have a tangent space $T_X\mathcal{M}$.

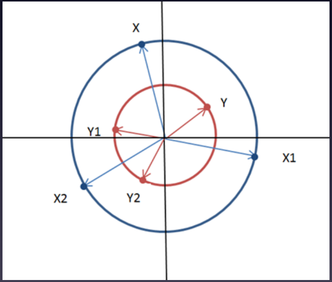
$\mathcal{M} = \mathcal{R}^2$
$T_X\mathcal{M} = \mathcal{R}^2$



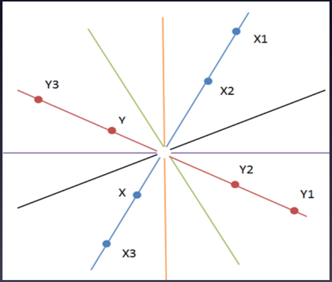
Quotient geometry: Equivalence classes / Fibers

Given a manifold $\mathcal{M}$, group G and an action, we get a set of equivalence classes / fibers on $\mathcal{M}$ by the action of G.

$\mathcal{M}_1 = \mathcal{R}^2$
$G = \{R \in \mathcal{R}^{2\times 2} \mid R^\top R = I_2\}$
Action: Left multiplication



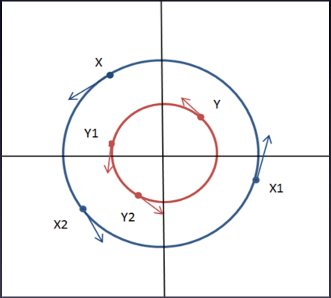
$\mathcal{M}_2 = \mathcal{R}^2 - \{0\}$
$G = \mathcal{R} - \{0\}$
Action: Scalar multiplication

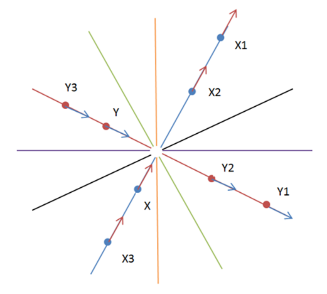


Quotient geometry: Vertical space

Vertical space $\mathcal{V}_X\mathcal{M}$ at $X \in \mathcal{M}$ is the set of all tangent vectors in $\mathcal{T}_X\mathcal{M}$ that are tangent to the equivalence class / fiber of X.

Vertical space can also be seen as the set of tangent vectors, movements along which keep X in its equivalence class / fiber.

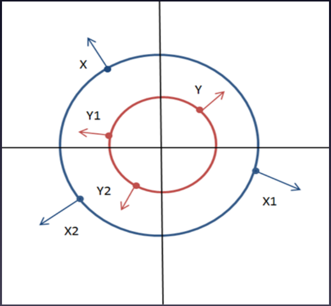


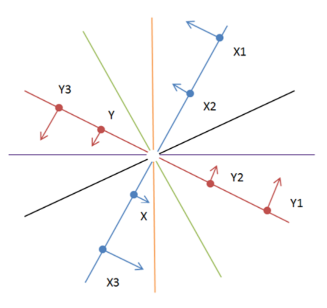


Quotient geometry: Horizontal space

Horizontal space $\mathcal{H}_X\mathcal{M}$ at $X \in \mathcal{M}$ is the orthogonal complement of vertical space $\mathcal{V}_X\mathcal{M}$.

Horizontal space can also be seen as the set of tangent vectors, movements along which move X across equivalence classes / fibers.





Quotient geometry: Horizontal space

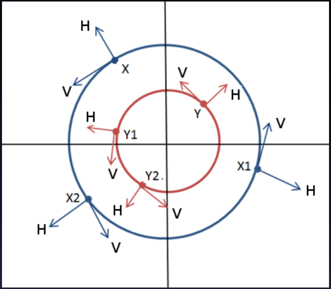
Horizontal vectors actually correspond to movements in the quotient space.

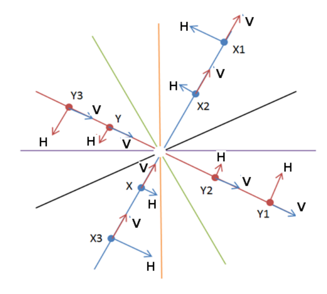
Horizontal vectors at a point $X \in \mathcal{M}$ provide a representation for the tangents vectors to the quotient space $\mathcal{M}/G$ at GX.

Dimensionality of quotient space is equal to the dimensionality of horizontal space.

Dimensionality of tangent space is equal to the sum of dimensionalities of Horizontal and Vertical spaces.

Quotient geometry: Horizontal and Vertical spaces





Space of circles in $\mathcal{R}^2$ centered at the origin is a one dimensional manifold.

Real projective space $\mathcal{RP}^1$ is a one dimensional manifold.

Quotient geometry: Horizontal and Vertical spaces

Exercise 1:

$G = \mathcal{O}_3 = \left\{ R \in \mathcal{R}^{3\times 3} \mid R^\top R = I_3 \right\}$

$\mathcal{M}_1 = \mathcal{R}^3$

Action: Left multiplication

Fibers:

Quotient geometry: Horizontal and Vertical spaces

Exercise 1:

$G = \mathcal{O}_3 = \left\{ R \in \mathcal{R}^{3\times 3} \mid R^\top R = I_3 \right\}$

$\mathcal{M}_1 = \mathcal{R}^3$

Action: Left multiplication

Fibers: Spheres centered at the origin in $\mathcal{R}^3$

Quotient geometry: Horizontal and Vertical spaces

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$G = \mathcal{O}_3 = \left\{ R \in \mathcal{R}^{3\times 3} \mid R^\top R = I_3 \right\}$

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Action: Left multiplication

Fibers: Spheres centered at the origin in $\mathcal{R}^3$

Tangent space at $X \in \mathcal{M}_1$:

Quotient geometry: Horizontal and Vertical spaces

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$G = \mathcal{O}_3 = \left\{ R \in \mathcal{R}^{3\times 3} \mid R^\top R = I_3 \right\}$

$\mathcal{M}_1 = \mathcal{R}^3$

Action: Left multiplication

Fibers: Spheres centered at the origin in $\mathcal{R}^3$

Tangent space at $X \in \mathcal{M}_1$: $\mathcal{R}^3$ (3-dimensional)

Quotient geometry: Horizontal and Vertical spaces

Exercise 1:

$G = \mathcal{O}_3 = \left\{R \in \mathcal{R}^{3\times 3} \mid R^\top R = I_3\right\}$

$\mathcal{M}_1 = \mathcal{R}^3$

Action: Left multiplication

Fibers: Spheres centered at the origin in $\mathcal{R}^3$

Tangent space at $X \in \mathcal{M}_1$: $\mathcal{R}^3$ (3-dimensional)

Vertical space at $X \in \mathcal{M}_1$:

Quotient geometry: Horizontal and Vertical spaces

Exercise 1:

$G = \mathcal{O}_3 = \left\{ R \in \mathcal{R}^{3\times 3} \mid R^\top R = I_3 \right\}$

$\mathcal{M}_1 = \mathcal{R}^3$

Action: Left multiplication

Fibers: Spheres centered at the origin in $\mathcal{R}^3$

Tangent space at $X \in \mathcal{M}_1$: $\mathcal{R}^3$ (3-dimensional)

Vertical space at $X \in \mathcal{M}_1$: Tangent plane to the sphere passing through X (2-dimensional).

Quotient geometry: Horizontal and Vertical spaces

Exercise 1:

$G = \mathcal{O}_3 = \left\{ R \in \mathcal{R}^{3\times 3} \mid R^\top R = I_3 \right\}$

$\mathcal{M}_1 = \mathcal{R}^3$

Action: Left multiplication

Fibers: Spheres centered at the origin in $\mathcal{R}^3$

Tangent space at $X \in \mathcal{M}_1$: $\mathcal{R}^3$ (3-dimensional)

Vertical space at $X \in \mathcal{M}_1$: Tangent plane to the sphere passing through X (2-dimensional).

Horizontal space at $X \in \mathcal{M}_1$:

Quotient geometry: Horizontal and Vertical spaces

Exercise 1:

$G = \mathcal{O}_3 = \left\{ R \in \mathcal{R}^{3\times 3} \mid R^\top R = I_3 \right\}$

$\mathcal{M}_1 = \mathcal{R}^3$

Action: Left multiplication

Fibers: Spheres centered at the origin in $\mathcal{R}^3$

Tangent space at $X \in \mathcal{M}_1$: $\mathcal{R}^3$ (3-dimensional)

Vertical space at $X \in \mathcal{M}_1$: Tangent plane to the sphere passing through X (2-dimensional).

Horizontal space at $X \in \mathcal{M}_1$: Normal to the sphere passing through X (1-dimensional).

Space of spheres in $\mathcal{R}^2$ centered at the origin is a one dimensional manifold.

Quotient geometry: Horizontal and Vertical spaces

Exercise 2:

$G = \mathcal{R} - \{0\}$

$\mathcal{M}_1 = \mathcal{R}^3 - \{0\}$

Action: Scalar multiplication

Fibers:

Quotient geometry: Horizontal and Vertical spaces

Exercise 2:

$G = \mathcal{R} - \{0\}$

$\mathcal{M}_1 = \mathcal{R}^3 - \{0\}$

Action: Scalar multiplication

Fibers: Lines passing through the origin in $\mathcal{R}^3$.

Quotient geometry: Horizontal and Vertical spaces

Exercise 2:

$G = \mathcal{R} - \{0\}$

$\mathcal{M}_1 = \mathcal{R}^3 - \{0\}$

Action: Scalar multiplication

Fibers: Lines passing through the origin in $\mathcal{R}^3$.

Tangent space at $X \in \mathcal{M}_1$:

Quotient geometry: Horizontal and Vertical spaces

Exercise 2:

$G = \mathcal{R} - \{0\}$

$\mathcal{M}_1 = \mathcal{R}^3 - \{0\}$

Action: Scalar multiplication

Fibers: Lines passing through the origin in $\mathcal{R}^3$.

Tangent space at $X \in \mathcal{M}_1$: $\mathcal{R}^3$ (3-dimensional)

Quotient geometry: Horizontal and Vertical spaces

Exercise 2:

$G = \mathcal{R} - \{0\}$

$\mathcal{M}_1 = \mathcal{R}^3 - \{0\}$

Action: Scalar multiplication

Fibers: Lines passing through the origin in $\mathcal{R}^3$.

Tangent space at $X \in \mathcal{M}_1$: $\mathcal{R}^3$ (3-dimensional)

Vertical space at $X \in \mathcal{M}_1$:

Quotient geometry: Horizontal and Vertical spaces

Exercise 2:

$G = \mathcal{R} - \{0\}$

$\mathcal{M}_1 = \mathcal{R}^3 - \{0\}$

Action: Scalar multiplication

Fibers: Lines passing through the origin in $\mathcal{R}^3$.

Tangent space at $X \in \mathcal{M}_1$: $\mathcal{R}^3$ (3-dimensional)

Vertical space at $X \in \mathcal{M}_1$: Line passing through origin and X (1-dimensional).

Quotient geometry: Horizontal and Vertical spaces

Exercise 2:

$G = \mathcal{R} - \{0\}$

$\mathcal{M}_1 = \mathcal{R}^3 - \{0\}$

Action: Scalar multiplication

Fibers: Lines passing through the origin in $\mathcal{R}^3$.

Tangent space at $X \in \mathcal{M}_1$: $\mathcal{R}^3$ (3-dimensional)

Vertical space at $X \in \mathcal{M}_1$: Line passing through origin and X (1-dimensional).

Horizontal space at $X \in \mathcal{M}_1$:

Quotient geometry: Horizontal and Vertical spaces

Exercise 2:

$G = \mathcal{R} - \{0\}$

$\mathcal{M}_1 = \mathcal{R}^3 - \{0\}$

Action: Scalar multiplication

Fibers: Lines passing through the origin in $\mathcal{R}^3$.

Tangent space at $X \in \mathcal{M}_1$: $\mathcal{R}^3$ (3-dimensional)

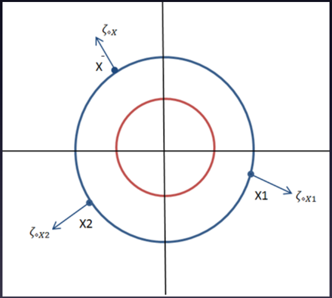
Vertical space at $X \in \mathcal{M}_1$: Line passing through origin and X (1-dimensional).

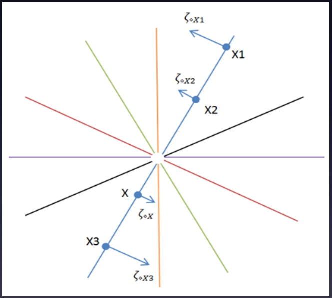
Horizontal space at $X \in \mathcal{M}_1$: Plane orthogonal to the line passing through origin and X (2-dimensional).

Space of lines passing through the origin in $\mathcal{R}^3$ is a two dimensional manifold.

Quotient geometry: Horizontal lift

Let ζ be a tangent vector to the quotient space $\mathcal{M}/G$ at GX. Then, the corresponding horizontal vector at $X \in \mathcal{M}$, denoted by $\zeta_{\diamond X} \in \mathcal{T}_X\mathcal{M}$ is referred to as horizontal lift of ζ at X.





Quotient geometry: Riemannian metric

We need to equip the quotient space $\mathcal{M}/G$ with a Riemannian metric to make it a Riemannian manifold.

We can derive a Riemannian metric for the quotient space $\mathcal{M}/G$ using the Riemannian metric of $\mathcal{M}$.

The idea is to define inner product between two tangent vectors ζ and η at $GX \in \mathcal{M}/G$ in terms of inner product between their horizontal lifts $\zeta_{\diamond X}$ and $\eta_{\diamond X}$ at $X \in \mathcal{M}$.

Since a single point $GX \in \mathcal{M}/G$ corresponds to multiple points in $\mathcal{M}$, we need to make sure that the value of the inner product between ζ and η at $GX \in \mathcal{M}/G$ does not depend on the $X \in \mathcal{M}$ used for lifting.

Quotient geometry: Riemannian metric

Space of circles: $< \zeta, \eta >_{GX} = \zeta_{\diamond X}^{\top} \eta_{\diamond X}$

Space of spheres: $< \zeta, \eta >_{GX} = \zeta_{\diamond X}^{\top} \eta_{\diamond X}$

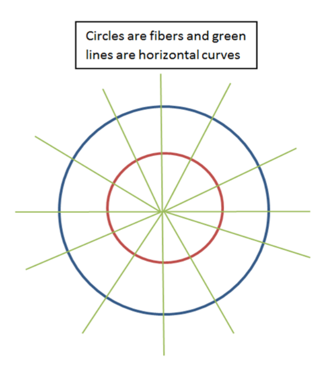
Real projective space $\mathcal{RP}^1$: $< \zeta, \eta >_{GX} = \frac{\zeta_{\diamond X}^{\top} \eta_{\diamond X}}{X^{\top} X}$

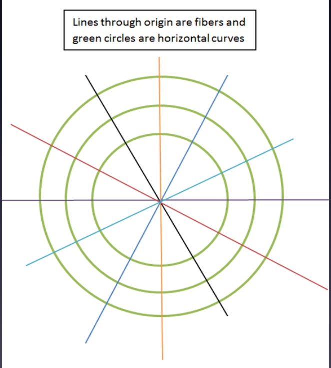
Real projective space $\mathcal{RP}^2$: $< \zeta, \eta >_{GX} = \frac{\zeta_{\diamond X}^{\top} \eta_{\diamond X}}{X^{\top} X}$

Quotient geometry: Horizontal curve

A horizontal curve $c(t) \in \mathcal{M}$ is a curve such that the tangent vector $c^{'}(t)$ is a horizontal vector for all t.

Horizontal curves on $\mathcal{M}$ correspond to curves in the quotient space $\mathcal{M}/G$.





Quotient geometry: Geodesics in quotient space

Tangent vectors on $\mathcal{M}/G \longrightarrow$ Horizontal vectors on $\mathcal{M}$.

Curves on $\mathcal{M}/G \longrightarrow$ Horizontal curves on $\mathcal{M}$.

Geodesics on $\mathcal{M}/G \longrightarrow$

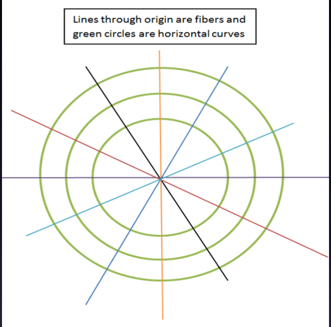
Quotient geometry: Geodesics in quotient space

Tangent vectors on $\mathcal{M}/G \longrightarrow$ Horizontal vectors on $\mathcal{M}$.

Curves on $\mathcal{M}/G \longrightarrow$ Horizontal curves on $\mathcal{M}$.

Geodesics on $\mathcal{M}/G \longrightarrow$ Horizontal geodesics on $\mathcal{M}$???

Quotient geometry: Geodesics in quotient space



It is not guaranteed that we will always have horizontal geodesics on $\mathcal{M}$.

In the above figure, horizontal curves are circles, which are not geodesics on $\mathcal{M}(\mathcal{R}^2 - \{0\})$.

What if we have horizontal geodesic curves on $\mathcal{M}$? Can we represent the geodesic curves on $\mathcal{M}/G$ using the horizontal geodesics on $\mathcal{M}$?

- Stiefel and Grassmann manifolds – Yes.
- General case – Not sure.

Quotient geometry: Summary

- Quotient space
- Horizontal space
- Vertical space
- Horizontal lift
- Quotient space Riemannian metric
- Horizontal curves
- Horizontal Geodesic curves

Quotient geometry: Take home message

If we know a manifold $\mathcal{M}_1$ very well, i.e., we have closed form expressions for geodesics, distances, parallel transport, etc., then we can derive closed form expressions for these quantities on quotient spaces of $\mathcal{M}$.